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A STUDY ON THE JACOBSON RADICAL OF A TERNARY Γ-SEMIRING

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ABSTRACT

In this paper we will study the Jacobson radical of a ternary Γ-semiring by using ternary Γ-semi modules. In section 2, we first give some preliminaries. In section 3, we will introduce and study the primitive ternary Γ-semiring. In section 4, we will study the Jacobson radical of a ternary Γ-semiring and the Jacobson semi simple ternary Γ-semiring

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Case Study

1. INTRODUCTION

The theory of ternary algebraic systems was studied by LEHMER [3] in 1932, but earlier such structures were investigated and studied by PRUFER [5] in 1924. In 1929 BAER [1] who gave the idea of n-ary algebras. In 2004, T.K. Dutta and S. Kar[2] were studied the Jacobson radical of a ternary semiring. In 2015, M. Sajani Lavanya, D. Madhusudhana Rao and V. Syam Julius Rajendra [6, 7, and 8] were investigated and studied about ternary
Γ-semiring. For notions and terminologies not given in this paper, the reader is referred to Sajani Lavanya, Madhusudhana Rao, and Syam Julius Rajendra [6, 7, and 8].

2. PRELIMINARIES

Definition 2.1(Sajani Lavanya, Madhusudana Rao and syam Julius Rajendra [7]): Let T and Γ be two additive commutative semi groups. T is said to be a ***Ternary*** **Γ*-semiring*** if there exist a mapping from T ×Γ× T ×Γ× T to T which maps $(x_1, \alpha, x_2, \beta, x_3) \to [x_1\alpha x_2\beta x_3]$ satisfying the conditions :

i) $[[a\alpha b\beta c]\gamma d\delta e] = [a\alpha[b\beta c\gamma d]\delta e] = [a\alpha b\beta[c\gamma d\delta e]]$

ii) $[(a + b)\ \alpha c\beta d] = [a\alpha c\beta d] + [b\alpha c\beta d]$

iii) $[a\alpha\ (b + c)\beta d] = [a\alpha b\beta d] + [a\alpha c\beta d]$

iv) $[a\alpha b\beta(c + d)] = [a\alpha b\beta c] + [a\alpha b\beta d]$ for all $a, b, c, d \in$ T and $\alpha, \beta, \gamma, \delta \in \Gamma$.

Definition 2.2: (Sajani Lavanya, Madhusudana Rao and syam Julius Rajendra [7]: A ternary Γ-semiring T is said to be ***commutative ternary*** **Γ*-semiring*** provided $a\Gamma b\Gamma c = b\Gamma c\Gamma a = c\Gamma a\Gamma b = b\Gamma a\Gamma c = c\Gamma b\Gamma a =$

$a\Gamma c\Gamma b$f or all $a, b, c \in$ T.

Definition 2.3: (Sajani Lavanya, Madhusudana Rao and syam Julius Rajendra [6]: An element 0of a ternary Γ-semiring T is said to be an ***absorbingzero*** of T provided $0 + x = x = x + 0$and $0\alpha a\beta b = a\alpha 0\beta b = a\alpha b\beta 0 = 0\ \forall\ a, b, x \in$ T and $\alpha, \beta \in \Gamma$.

Note 2.4. Throughout this paper, T will always denote a ternary Γ-semiring with zero and unless otherwise stated a ternary Γ-semiring means a ternary Γ-semiring with zero.

Definition 2.5: (Sajani Lavanya, Madhusudana Rao and syam Julius Rajendra [7]: An element a_i of a ternary Γ-semiring T is said to be an ***identity*** provided $\sum_{i=1}^{n} a_i\alpha_i a_i\beta_i t = \sum_{i=1}^{n} a_i\alpha_i t\beta_i a_i = \sum_{i=1}^{n} t\alpha_i a_i\beta_i a_i = t\ \forall\ t \in$ T, $\alpha_i, \beta_i \in \Gamma$. In this case the ternary Γ-semiring is said to be a ternary Γ-semiring with identity.

Definition 2.6: (Sajani Lavanya, Madhusudana Rao and syam Julius Rajendra [8]: Let T be ternary Γ-semiring. A non empty subset 'S' is said to be a ***ternary subΓ-semiring*** of T if S is an additive subsemigroup of T and $a\alpha b\beta c \in$ S for all $a, b, c \in$ S and $\alpha, \beta \in \Gamma$.

Definition 2.7: (Sajani Lavanya, Madhusudana Rao and syam Julius Rajendra [8: A nonempty subset A of a ternary Γ-semiring T is said to be ***ternary Γ-ideal*** of T if

(1) $a, b \in$ A $\Rightarrow a + b \in$ A

(2) $b, c \in$ T , $a \in$ A, $\alpha, \beta \in \Gamma \Rightarrow b\alpha c\beta a \in$ A, $b\alpha a\beta c \in$ A, $a\alpha b\beta c \in$ A.

Definition 2.8: (Dutta. T. K. and **Kar. S [2]):** A ternary Γ-ideal I of T is said to be a ***k-ternary Γ-ideal*** if $x + y \in$ I, $x \in$ T, $y \in$I implies that $x \in$ I.

Definition 2.9: (Dutta. T. K. and **Kar. S [2]):** A ternary Γ-ideal I of T is said to be a ***h-ternary Γ-ideal*** provided $x + y_1 + z = y_1 + z$; $x, z \in$ T and $y_1, y_2 \in$ I implies that $x \in$ I.

Clearly, every *h*-ternary Γ-ideal is a *k*-ternary Γ-ideal of T and the intersection of an arbitrary collection of *h*-ternary Γ-ideals is again an *h*-ternary Γ-ideal of T.

Let A be a ternary Γ-ideal of T. Then the ***k-closure*** of A, denoted by $\overline{A}$, is defined by $\overline{A} = \{a \in T : a + b = c \text{ for some } b, c \in A\}$. We note that a ternary Γ-ideal A of S is a *k*-ternary Γ-ideal if and only if $A = \overline{A}$.

3. PRIMITIVE TERNARY Γ-SEMIRING

Definition 3.1: An equivalence relation ρ on T is said to be a ***ternary Γ-congruence relation*** or simply a ***Γ-congruence*** of T if the following conditions are satisfied:

(i) $a\rho a'$ And $b\rho b' \Rightarrow (a+b)\rho(a'+b')$ as well as

(ii) $a\rho a', b\rho b'$ and $c\rho c' \Rightarrow (a\alpha b\beta c)\rho(a'\alpha b'\beta c')$ For all $a, a', b, b', c, c' \in T, \alpha, \beta \in \Gamma$.

The condition (ii) of the above definition is equivalent to the following condition:

(ii) $a\rho a' \Rightarrow (a\alpha b\beta c)\rho(a'\alpha b\beta c), (b\alpha a\beta c)\rho(b\alpha a'\beta c), (b\alpha c\beta a)\rho(b\alpha c\beta a')$.

Definition 3.2: Let A be a proper ternary Γ-ideal of T. Then the Γ-congruence on T , denoted by ρ_I and defined by $t\rho t'$ if and only if $t+a_1 = t'+a_2$ for some $a_1, a_2 \in A$, is called the ***Bourne Ternary Γ-Congruence*** on T defined by the ternary Γ-ideal A.

We denote the Bourne ternary Γ-congruence (ρ_I) class of an element *t* of T by t/ρ_I or simply by t/A and denote the set of all such ternary Γ-congruence classes of T by T/ρ_I or simply by T/A . We observe here that for any $s \in$ T and for any proper ternary Γ-ideal A of T, $s/A \in T/A$ is not necessarily equal to s + I = {$s + a : a \in$ I}.

Definition 3.3: For any proper ternary Γ-ideal of T if the Bourne ternary Γ-congruence ρ_I, defined by A, is proper i.e. $0/A \neq T$, then we can define the operations, addition and ternary multiplication on T/A by $a/A + b/A = (a+b)/A$ and $(a/A)\alpha(b/A)\beta(c/A) = (a\alpha b\beta c)/A$ for all *a, b, c*∈ T, *α, β*∈Γ. With these two operations, we see that T/A is a ternary Γ-semiring and we call this ternary Γ-semiring the ***Bourne factor ternary Γ-semiring*** or simply the ***factor ternary Γ-semiring***.

Definition 3.4: Let S and T be two ternary Γ-semirings. Let *f* be a mapping which maps from S to T. Then *f* is said to be a ***ternary Γ-homomorphism*** of S into T if

(i) $f(x+y) = f(x)+f(y)$ And

(ii) $f(a\alpha b\beta c) = f(a)\alpha f(b)\beta f(c)$ For all $a, b, c \in T, \alpha, \beta \in \Gamma$.

If *f* is both one-one and onto then *f* is called a ***Γ-isomorphism***

Definition 3.5: An additive commutative semigroup M with a zero element 0_M is said to be a ***right ternary TΓ-semimodule*** if there exist a mapping $M \times \Gamma \times T \times \Gamma \times T \to M$, denoted by $(x, \alpha, a, \beta, b) \to x\alpha a\beta b$, which satisfies the following conditions for all elements $x, y \in M$, $a, b, c, d \in T, \alpha, \beta, \gamma, \delta \in \Gamma$:

$$
\begin{aligned}
&(i)\ (x+y)\alpha a\beta b = x\alpha a\beta b + y\alpha a\beta b\\
&(ii)\ x\alpha a\beta(b+c) = x\alpha a\beta b + x\alpha a\beta c\\
&(iii)\ x\alpha(a+b)\beta c = x\alpha a\beta c + x\alpha b\beta c\\
&(iv)\ (x\alpha a\beta b)\gamma c\delta d = x\alpha(a\beta b\gamma c)\delta d = x\alpha a\beta(b\gamma c\delta d)\\
&(v)\ 0_M\alpha a\beta b = 0_M = x\alpha a\beta 0_T = x\alpha 0_T\beta b.
\end{aligned}
$$

In addition to the above conditions if $\sum_{i=1}^{n} m\alpha a_i\beta a_i = m$ holds for all *m*∈ M, where a_i is an identity element of T, then M is said to be a ***unitary right ternary TΓ-semimodule***.

Similarly, a left ternary TΓ-semimodule can be defined.

Example 3.6: Every ternary Γ-semiring T is a right ternary TΓ-semimodule under the right ternary multiplication

in the ternary Γ-semiring T.

Example 3.7: Let $M_2(Z^-)$ be the ternary Γ-semiring of all 2×2 square matrices over Z^-, the set of all negative integers. Then $I_2 = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in Z \right\}$ forms a right ternary TΓ-semimodule over $M_2(Z^-)$

Example 3.7: Let D be a division ternary Γ-semiring. Let $M_{p,q}(D)$ denote the additive semigroup of all p×q matrices whose entries are form D and D_p be the set of all p-tuples of elements of D. Then D_p as well as $M_{p,q}(D)$ can be made in natural way into TΓ-semimodule for $\Gamma = M_{p,q}(D)$ and $T = M_{q,p}(D)$.

Definition 3.8: A nonempty subset N of a right ternary TΓ-semimodule M is said to be a ***ternary sub TΓ-semimodule*** of M provided (i) $a + b \in$ N, (ii) $a\alpha s\beta t \in$ N, (iii) N contains the zero of M for all a, $b \in$ N, s, $t \in$ T and for all α, $\beta \in \Gamma$.

Most of the results on a ternary semiring S can be established for a right ternary S-semimodule M with some mild modifications. For example, every ternary *h*-sub semimodule is a *k*-sub semimodule of M.

Definition 3.9: Let M and N be two right ternary TΓ-semimodules and ψ a mapping from M into N. Then ψ is said to be a ***TΓ-homomorphism*** of M into N if $\psi(a + b) = \psi(a) + \psi(b)$ and $\psi(a\alpha s\beta t) = \psi(a)\alpha s\beta t$ for all a, $b \in$ M, s, $t \in$ T and α, $\beta \in \Gamma$.

Definition 3.10: A right ternary TΓ-semimodule M is said to be ***additively cancellative*** if $a + b = a + c$ implies that $b = c$ for all a, b, $c \in$ M. In this case M is called ***additively cancellative right ternary TΓ-semimodule***. Similarly, we can define additively cancellative ternary Γ-semiring.

Note 3.11: In an additively cancellative ternary Γ-semiring the concept of *h*-ternary Γ-ideal and *k*-ternary Γ-ideal coincide.

Definition 3.12: The ***zeroid*** of a ternary Γ-semiring T, denoted by Z (T), is defined as $Z(T) = \{x \in T : x + z = z \text{ for some } z \in T\}$. Clearly, the zero element 0_T of T is a member of Z (T).

Lemma 3.13: The zeroid Z (T) of a ternary Γ-semiring T is an *h*-ternary Γ-ideal of T.

***Proof*:** Let $t_1, t_2 \in Z(T)$ then $t_1 + t_2 = r_1$ and $t_2 + r_2 = r_2$ for some $r_1, r_2 \in T$

$\Rightarrow t_1 + t_2 + r_1 + r_2 = r_1 + r_2$, since addition is commutative and hence $t_1 + t_2 \in Z(T)$.

Let $s,\ t \in T,\ \alpha,\ \beta \in \Gamma$, then $t_1\alpha s\beta t + r_1\alpha s\beta t = (t_1 + r_1)\alpha s\beta t = r_1\alpha s\beta t$ and so $r_1\alpha s\beta t \in Z(T)$ Hence Z (T) is a right ternary Γ-ideal of T.

In a similar manner we can prove Z(T) is a left ternary Γ-ideal as well as lateral ternary Γ-ideal of T. Therefore Z (T) is a ternary Γ-ideal of T.

Suppose that $r + s_1 + t = s_2 + t;$ where $r, t \in T$ and $s_1, s_2 \in Z(T)$.

Since $s_1, s_2 \in Z(T), s_1 + t_1 = t_1$ and $s_2 + t_2 = t_2$

Now $r + s_1 + t = s_2 + t \Rightarrow r + s_1 + t_1 + t + t_2 = s_2 + t_2 + t_1 + t$

$\Rightarrow r + t_1 + t + t_2 = t_2 + t + t_1 = t_1 + t + t_2 \Rightarrow r \in Z(T)$.

Therefore Z (T) is an *h*-ternary Γ-ideal of T.

Remark 3.14: The zeroid of a ternary Γ-semiring T is the smallest *h*-ternary Γ-ideal of T.

Definition 3.15: Let M be a right ternary TΓ-semimodule.

We put $(0:M) = \{x \in T : m\Gamma s\Gamma x = 0 \text{ and } m\Gamma x\Gamma s = 0 \ \forall m \in M \text{ and } \forall s \in T\}$.

Then we call (0: M) the ***annihilator*** of M in T, denoted by $A_T(M)$.

Note 3.16: The zeroid Z (T) of T is contained in $A_T(M)$.

Lemma 3.17: $A_T(M)$ is an *h*-ternary Γ-ideal of T.

***Proof*:** Clearly, $A_T(M)$ is an additive sub semigroup of T. Suppose $x \in A_T(M)$, then $m\Gamma s\Gamma x = 0$ and $m\Gamma x\Gamma s = 0$ for all $m \in M, s \in T$ and $\alpha, \beta \in \Gamma$. Now for all $m \in M, r, s, t \in T$ and $\alpha, \beta, \gamma, \delta \in \Gamma$, $m\Gamma r\Gamma(x\Gamma s\Gamma t) = (m\Gamma r\Gamma x)\Gamma s\Gamma t = 0$

And $m\Gamma(x\Gamma s\Gamma t)\Gamma r = (m\Gamma x\Gamma s)\Gamma t\Gamma r = 0$. Thus $x\Gamma s\Gamma t \subseteq A_T(M)$ for all $s, t \in T$

Similarly, we can show that $s\Gamma t\Gamma x \subseteq A_T(M)$ and $s\Gamma x\Gamma t \subseteq A_T(M)$ for all *s*, *t* ∈ T.

Hence $A_T(M)$ is a ternary Γ-ideal of T.

We now show that $A_T(M)$ is an *h*-ternary Γ-ideal of T.

For this purpose, we let $x + t_1 + y = t_2 + y$, where $x, y \in T$ and $t_1, t_2 \in A_T(M)$.

Since $t_1, t_2 \in A_T(M)$, $m\Gamma t\Gamma t_1 = m\Gamma t_1\Gamma t = 0$ and $m\Gamma t\Gamma t_2 = m\Gamma t_2\Gamma t = 0$

For all $m \in M$ and for all $t \in T$.

Now $x + t_1 + y = t_2 + y \Rightarrow m\Gamma t\Gamma x + m\Gamma t\Gamma t_1 + m\Gamma t\Gamma y = m\Gamma t\Gamma t_2 + m\Gamma t\Gamma y$

This leads to $m\Gamma t\Gamma x = 0$, since $m\Gamma t\Gamma t_1 = m\Gamma t\Gamma t_2 = 0$ and M is additively cancellative. Similarly, we can show that $m\Gamma x\Gamma t = 0$ for all $m \in M$ and for all $x, t \in T$.

Thus $x \in A_T(M)$ and hence $A_T(M)$ is an *h*-ternary Γ-ideal of T.

Remark 3.18: Since every *h*-ternary Γ-ideal is a *k*-ternary Γ-ideal of T.

Definition 3.19: A right ternary TΓ-semimodule M is said to be ***faithful*** if $Z(T) = A_T(M)$.

Definition 3.20: A right ternary TΓ-semimodule M ≠ {0} is said to be ***irreducible*** if for every arbitrary fixed pair $u_1, u_2 \in M$ with $u_1 \neq u_2$ and for any $x \in M$ there exist $\alpha_1, \alpha_2, ..., \alpha_n, \beta_1, \beta_2, ...\beta_m, \gamma_1, \gamma_2, ..., \gamma_n, \delta_1, \delta_2, ..., \delta_m \in \Gamma$ and $a_1, a_2, ...a_n, b_1, b_2, ..., b_m,$

$c_1, c_2, ..., c_n, d_1, d_2, ..., d_m \in T$ Such that

$$x + \sum_{i=1}^{n} u_1 \alpha_i a_i \beta_i b_i + \sum_{j=1}^{m} u_2 \gamma_j c_j \delta_j d_j = \sum_{i=1}^{n} u_1 \gamma_j c_j \delta_j d_j + \sum_{j=1}^{m} u_2 \alpha_i a_i \beta_i b_i .$$

Lemma 3.21: Let I be an *h*-ternary Γ-ideal of a ternary Γ-semiring T. If M is an irreducible right ternary $(T/I)\Gamma$ -semimodule then M is an irreducible right ternary TΓ-semimodule.

***Proof*:** Suppose that M is an irreducible right ternary $(T/I)\Gamma -$ semimodule. Then we can define a ternary Γ-action on M by $m\Gamma s\Gamma t = m\Gamma(s/I)\Gamma(t/I)$ for all *m*∈ M and for all *s*, *t*∈ T, and this makes M into an irreducible right ternary TΓ-semimodule.

The converse of the lemma 3.21 is not necessarily true. But in particular we have the following theorem.

Theorem 3.22: If M is an irreducible right ternary TΓ-semimodule then M is an irreducible right ternary $(T/A_T(M))\Gamma$-semimodule, where $T/A_T(M)$ is a factor ternary Γ-semiring.

***Proof*:** Suppose M is an irreducible right ternary TΓ-semimodule. We define a ternary Γ-action on M as follows: $m\Gamma(s/I)\Gamma(t/I) = m\Gamma s\Gamma t$ where I = $A_T(M)$, for all *m*∈ M and for all *s*, *t*∈ T.

We now show that the above definition is well defined. If $t/A_T(M) = t'/A_T(M)$ then $t + i_1 + z_1 = t' + i_2 + z_1$ for some $i_1, i_2 \in A_T(M)$ and $z_1 \in T$

Since $i_1, i_2 \in A_T(M)$, we have $m\Gamma s\Gamma i_1 = m\Gamma s\Gamma i_2 = 0.$

Now $t + i_1 + z_1 = t' + i_2 + z_1 \Rightarrow m\Gamma s\Gamma t + m\Gamma s\Gamma i_1 + m\Gamma s\Gamma z_1 = m\Gamma s\Gamma t' + m\Gamma s\Gamma i_2 + m\Gamma s\Gamma z_1$ for all *m*∈ M and *s*∈ T which implies that $m\Gamma s\Gamma t = m\Gamma s\Gamma t'$ $\rightarrow (1)$

Again if $s/A_T(M) = s'/A_T(M)$ then $s + i_3 + z_2 = s' + i_4 + z_2$ for some $i_3, i_4 \in A_T(M)$ and $z_2 \in T.$ Since $i_3, i_4 \in A_T(M)$, so $m\Gamma i_3\Gamma t' = m\Gamma i_4\Gamma t' = 0.$ Also $0 + i_3 + z_2 = s' + i_4 + z_2$

$\Rightarrow m\Gamma s\Gamma t' + m\Gamma i_3\Gamma t' + m\Gamma z_2\Gamma t' = m\Gamma s'\Gamma t' + m\Gamma i_4\Gamma t' + m\Gamma z_2\Gamma t'$ for all *m*∈ M and $t' \in T$ which implies that $m\Gamma s\Gamma t' = m\Gamma s'\Gamma t'$ $\rightarrow (2)$

From (1) and (2), it follows that $m\Gamma s\Gamma t = m\Gamma s'\Gamma t'$.

Thus $m\Gamma(s/A_T(M))\Gamma(t/A_T(M)) = m\Gamma(s'/A_T(M))\Gamma(t'/A_T(M)) \Rightarrow m\Gamma s\Gamma t = m\Gamma s'\Gamma t'$ and hence the above definition is well defined. Now it is easy to see that the above definition makes M into an irreducible right ternary $(T/A_T(M))\Gamma$-semimodule.

Lemma 3.23: A right ternary TΓ-semimodule M is a faithful (T/A_T (M)) Γ-semimodule.

***Proof*:** To prove M is faithful we need to show that $A_{T/A_T(M)}(M) = Z\Gamma(T/A_T(M))$.

From note 3.16, we see that $Z\Gamma(T/A_T(M)) \subseteq A_{T/A_T(M)}(M)$.

For the converse part, we let $x/A_T(M) \in A_{T/A_T(M)}(M)$.

Then $m\Gamma(t/A_T(M))\Gamma(x/A_T(M)) = 0$ and $m\Gamma(x/A_T(M))\Gamma(t/A_T(M)) = 0$

i. e. $m\Gamma t\Gamma x = 0$ and $m\Gamma x\Gamma t = 0$ for all $m\in$ M and for all $t\in$ T

Thus $x \in A_T(M)$ and hence $x/A_T(M) = 0/A_T(M)$.

Consequently, $x/A_T(M) \in Z\Gamma(T/A_T(M))$ and so $A_{T/A_T(M)}(M) \subseteq Z\Gamma(T/A_T(M))$.

Thus $A_{T/A_T(M)}(M) = Z\Gamma(T/A_T(M))$. Hence the lemma is proved.

Lemma 3.24: If P is an *h*-ternary Γ-ideal of a ternary Γ-semiring T, then $Z\Gamma(T/P) = \{0\}$ where T/P is a factor ternary Γ-semiring.

***Proof*:** Suppose *s*/P $\in Z\Gamma(T/P)$. Then we have *s/P* + *t/P* = *t/P* for some *t/P*∈ T/P. This implies that (*s* + *t*)/P = *t/P* which implies that $s+t+i_1 = t_1 + t_2$ for some $i_1, i_2 \in P$. this shows that *s*∈ P, since P is an *h*-ternary Γ-ideal of T. Consequently, *s*/P = 0/P. Thus $Z\Gamma(T/P) = \{0\}$.

Definition 3.25: A ternary Γ-semiring T is said to be ***primitive*** if it has a faithful irreducible ternary TΓ-semimodule. A ternary Γ-ideal P is said to be ***primitive*** if the factor ternary Γ-semiring T/P is primitive. Hence a ternary Γ-semiring T is primitive if {0} is a primitive ternary Γ-ideal of T.

The following is a characterization theorem for primitive ternary Γ-ideal of ternary Γ-semirings.

Theorem 3.26: An h-ternary Γ-ideal P of a ternary Γ-semiring T is primitive if and only if P = $A_T(M)$ for some irreducible right ternary TΓ-semimodule M.

***Proof*:** Let *P* be an h-ternary Γ-ideal of *T* such that *P* = $A_T(M)$ for some irreducible right ternary *T*Γ-semimodule *M*. Then by theorem 3.22 and Lemma 3.23*M*is a faithful irreducible ternary (*T/P*) Γ-semimodule this shows that *T/P* is primitive and hence *P* is a primitive h-ternary Γ-ideal of *T.*

Conversely, let *P* be a primitive h-ternary Γ-ideal of *T*. Then T/*P* is a primitive ternary Γ-semiring. So there exists a faithful irreducible ternary (*T/P*) Γ-semimodule *M*. Now by Lemma 3.21*M* is an irreducible ternary *T*Γ-semimodule. It remains to show that *P* = $A_T(M)$. Now $x \in A_T(\mathrm{M}) \Leftrightarrow x \in T$ such that *m*Γ*s*Γ*x* = 0 and *m*Γ*x*Γ*s* = 0 for all *m* ∈ *M* and *s* ∈ *T*

⇔ *x/P* ∈ *T/P* such that *m*Γ(*s/P*)Γ(*x/P*) = 0 and *m*Γ(*x/P*)Γ(*s/P*) = 0for all *m* ∈ *M* and *s/P* ∈ *S/P*⇔ *x/P* $\in A_{T/P}(M) = Z\Gamma(T/P)$, since *M* is a faithful ternary (*T/P*)Γ-semimodule ⇔ *x/P* ∈ $A_{T/P}(M) = \{0\}$, by Lemma3.24,⇔ *x/P* = 0*/P*⇔ *x* ∈ *P*. Thus *P* = $A_T(M)$. Hence the lemma

4. JACOBSON RADICAL OF A TERNARY Γ-SEMIRING

In the previous section, we have defined irreducible ternary TΓ-semimodule. We now we give the definition of semi-irreducible ternary TΓ-semimodule.

Definition 4.1: A right ternary TΓ-semimodule M is said to be ***semi-irreducible*** if MΓTΓT ≠ {0}. i. e. $\sum_{i=}^{n} m_i\alpha_i s_i\beta_i t_i \neq 0,$ where $m_i \in M$, $s_i, t_i \in T$ and $\alpha_i, \beta_i \in \Gamma$, and M does not contain any ternary *k*-sub semimodule other than {0} and M.

Theorem 4.2: Let A be an *h*-ternary Γ-ideal of a ternary Γ-semiring T and M a right ternary TΓ-semimodule with MΓTΓA ≠ {0}. Then the following statements are true:

1) If M is semi-irreducible and *m* is an element of M then *m* = 0 if and only if *m*Γ*t*Γ*a* = 0 for all *t*∈ T and for all *a*∈ A. i.e. *m* = 0 if and only if *m*ΓTΓA = {0}.

2) If M is irreducible and *u*, *v* are elements of M, then *u* = *v* if and only if $\sum_{i=1}^{m} u\Gamma a_i\Gamma b_i = \sum_{i=1}^{m} v\Gamma a_i\Gamma b_i$ **for all** $a_i, b_i \in T.$

Proof: (1) Let M be a semi-irreducible right ternary TΓ-semimodule and *m*Γ*t*Γ*a* = 0 for all *t*∈ T and for all *a*∈ A. Let

M_0 = {*y*∈ M; *y*ΓTΓA = {0} i. e. $\sum_{i=1}^{n} y\alpha_i s_i\beta_i a_i = 0, s_i \in T, a_i \in A, \alpha_i, \beta_i \in \Gamma\}.$

Then $m \in M_0$ and so M_0 is non-empty

Let *x*,*y*∈ M_0. Then $(x+y)\Gamma T\Gamma A = x\Gamma T\Gamma A + y\Gamma T\Gamma A = \{0\}.$

This leads to *x* + *y*∈ M_0. Now let *x*∈ M_0 and *s*, *t*∈ T. Then we get

$(x\Gamma s\Gamma t)\Gamma T\Gamma A \subseteq M_0\Gamma T\Gamma T\Gamma T\Gamma A \subseteq M_0\Gamma T\Gamma A = \{0\}$ i. e. $(x\Gamma s\Gamma t)\Gamma T\Gamma A = \{0\}.$

This implies that $x\Gamma s\Gamma t \subseteq M_0$ and therefore, M_0 is a ternary Γ-sub semimodule of M.

Again suppose $x+y\in M_0, y\in M_0$ and $x\in$ M. Then

$$\sum_{i=1}^{n}(x+y)\Gamma s_i\Gamma a_i=0,\sum_{i=1}^{n}y\Gamma s_i\Gamma a_i=0 \text{ for all } s_i\in T, a_i\in A.$$

$$\Rightarrow\sum_{i=1}^{n}x\Gamma s_i\Gamma a_i=\sum_{i=1}^{n}x\Gamma s_i\Gamma a_i+0=\sum_{i=1}^{n}x\Gamma s_i\Gamma a_i+\sum_{i=1}^{n}y\Gamma s_i\Gamma a_i=\sum_{i=1}^{n}(x+y)\Gamma s_i\Gamma a_i=0 \text{ so } x\in M_0.$$

This shows that M_0 is a ternary *k*-sub semimodule of M. Since MΓTΓA ≠ {0}, $M_0\neq$ M Again since M is semi-irreducible, M_0 = {0} and there by *m* =0.

The converse part is obvious.

2) Let M be irreducible and $u,v\in M$ be such that *u*≠*v*. Since MΓTΓA ≠ {0}, there exist *m*∈ M, *t*∈ T and *a*∈ A such that *m*Γ*t*Γ*a* ≠ 0. Again since M is irreducible, for this *m*, there exist $a_i,b_i,c_j,d_j\in T,\alpha_i\beta_i,\alpha_j,\beta_j\in\Gamma(1\le i\le p,1\le j\le q; p,q$ are positive integers) such that

$$m+\sum_{i=1}^{p}u\alpha_i a_i\beta_i b_i+\sum_{j=1}^{q}v\alpha_j c_j\beta_j d_j=\sum_{j=1}^{q}u\alpha_j c_j\beta_j d_j+\sum_{i=1}^{p}v\alpha_i a_i\beta_i b_i.$$

Hence $m\Gamma t\Gamma a+\sum_{i=1}^{p}u\alpha_i a_i\beta_i b_i\gamma t\delta a+\sum_{j=1}^{q}v\alpha_j c_j\beta_j d_j\lambda t\mu a=\sum_{j=1}^{q}u\alpha_j c_j\beta_j d_j\lambda t\mu a+\sum_{i=1}^{p}v\alpha_i a_i\beta_i b_i\gamma t\delta a$ for all $t\in T$ and $a\in A$.

This implies that $m\Gamma t\Gamma a+\sum_{i=1}^{p}u\Gamma a_i\Gamma b_i'+\sum_{j=1}^{q}v\Gamma c_j\Gamma d_j'=\sum_{j=1}^{q}u\Gamma c_j\Gamma d_j'+\sum_{i=1}^{p}v\Gamma a_i\Gamma b_i'.$

Where $b_i'=b_i\alpha t\beta a\in A$ and $d_j'=d_j\gamma t\delta a\in A$. Since M is cancellative and $m\Gamma t\Gamma a\neq 0$ so at least one of $\sum_{i=1}^{p}u\Gamma a_i\Gamma b_i'\neq\sum_{i=1}^{p}v\Gamma a_i\Gamma b_i'$ and $\sum_{j=1}^{q}u\Gamma c_j\Gamma d_j'\neq\sum_{j=1}^{q}v\Gamma c_j\Gamma d_j'$ holds.

The converse part follows easily.

Lemma 4.3: Let M be a right ternary TΓ-semimodule and $M\neq 0$. Then M is semi-irreducible if and only if for every nonzero $m\in M$, $\overline{m\Gamma T\Gamma T}=M$ i.e. for every arbitrary fixed nonzero $m\in M$ and every $x\in M$, there exist $a_i,b_i,c_j,d_j\in T$ such that $x+\sum_{i=1}^{p}m\Gamma a_i\Gamma b_i=\sum_{j=1}^{q}m\Gamma c_j\Gamma d_j$ where p, q are positive integers.

Proof: Let $M\neq 0$ be semi-irreducible. Then $M\Gamma T\Gamma T\neq\{0\}$

Let $m\in M$ such that $m\neq 0$. Then by theorem 4.2, $m\Gamma T\Gamma T\neq\{0\}$

Since $\overline{m\Gamma T\Gamma T}$ is a ternary *k*-subsemimodule of M, $\overline{m\Gamma T\Gamma T}=M$.

Conversely suppose that for any non-zero $m \in M$, $\overline{m\Gamma T\Gamma T} = M$.

Let $N \neq \{0\}$ be a ternary *k*-subsemimodule of M. Then there exist $n \in N$ such that $n \neq 0$. Therefore by hypothesis, $\overline{n\Gamma T\Gamma T} = M$

Hence for any $x \in M$, there exist $a_i, b_i, c_j, d_j \in T$ such that $x + \sum_{i=1}^{p} n\Gamma a_i \Gamma b_i = \sum_{j=1}^{q} n\Gamma c_j \Gamma d_j$. Since N is a ternary *k*-subsemimodule of M and $\sum_{i=1}^{p} n\Gamma a_i \Gamma b_i, \sum_{j=1}^{q} n\Gamma c_j \Gamma d_j \in N$, so we find that $x \in N$. Hence N = M. Now if $M\Gamma T\Gamma T = \{0\}$ then $m\Gamma T\Gamma T = \{0\}$ for all $m \in M$

In particular, $m\Gamma T\Gamma T = \{0\}$ for any non-zero $m \in M$ Hence $\overline{m\Gamma T\Gamma T} = \{0\}$ for any non-zero $m \in M$ this shows that M = {0}, which is a contradiction.

Thus $M\Gamma T\Gamma T \neq \{0\}$ and hence M is semi-irreducible.

Corollary 4.4: If a right ternary TΓ-semimodule M is irreducible then it is semi-irreducible and $\overline{M\Gamma T\Gamma T} = M$.

***Proof*:** Let M be an irreducible right ternary TΓ-semimodule. Then $M \neq 0$, and consequently, there exists a non-zero $m \in M$ .Since M is irreducible, for any arbitrary fixed $m \neq 0$ and any $x \in M$ there exist $a_i, b_i, c_j, d_j \in T, \alpha_i \beta_i, \alpha_j, \beta_j \in \Gamma(1 \leq i \leq p, 1 \leq j \leq q; p, q$ are positive integers) such that $x + \sum_{i=1}^{p} m\alpha_i a_i \beta_i b_i = \sum_{j=1}^{q} m\alpha_j c_j \beta_j d_j$ (From the definition of irreducibility, putting $u_1 = m$ and $u_2 = 0$).

Hence by lemma 4.3, *M* becomes a semi-irreducible right ternary TΓ-semimodule. Then $M\Gamma T\Gamma T \neq \{0\}$ this implies that $\overline{M\Gamma T\Gamma T} \neq \{0\}$. Since $\overline{M\Gamma T\Gamma T}$ is a ternary *K*-subsemimodule of *M*, $\overline{M\Gamma T\Gamma T}$ = *M*.

Definition 4.5: Let T be a ternary Γ-semiring and Δ be the set of all irreducible right ternary TΓ-semimodules. Then $J(T) = \bigcap_{M \in \Delta} A_T(M)$ is called the ***Jacobson radical*** of T

If Δ is empty the T itself is considered as J (T) i.e. *J* (T) = T and in this case, we say that T is a radical ternary Γ-semiring.

A ternary Γ-semiring T is said to be Jacobson semisimple or J-semisimple if J (T) = $\{0\}$.

Remark 4.6: The zeroid *Z* (*T*) of *T* is contained in the Jacobson radical *J* (*T*),

Since $Z(T) \subseteq A_T(M)$ for all right ternary TΓ-semimodule *M* by Note 3.16

Theorem 4.7: J (T) is an h-ternary Γ-ideal of T.

Proof**:** Since by Lemma 3.17, $A_T(M)$ is an h-ternary Γ-ideal of *T* and the intersection of any family of h-ternary Γ-ideals is again a h-ternary Γ-ideal, it follows that *J*(*T*) is an h-ternary Γ-ideal of *T*.

Corollary 4.8: *J* (*T*) *is a k*-ternary Γ-ideal *of T.*

Proof**:** The proof of the corollary immediately follows from the above theorem 4.7, since every h-ternary Γ-ideal is also a *k*-ternary Γ-ideal.

Theorem 4.9: The Jacobson radical of T is the intersection of all primitive h-ternary Γ-ideals of S.

Proof: The proof of the above theorem follows from theorem 3.26, and definition 4.5.

Definition4.10: Let P be a ternary Γ-ideal of T. Then P is said to be ***strongly semi-nilpotent*** if there exists a positive integer *n* such that $(P\Gamma T\Gamma)^{n-1}P \subseteq Z(T)$, where $(P\Gamma T\Gamma)^{n-1}P = (P\Gamma T)\Gamma(P\Gamma T).....(n-1)\Gamma P$ times, $(P\Gamma T\Gamma)^0 P = P$ and Z(T) is the zeroid of T. P is said to be strongly nilpotent if there exists a positive integer *n* such that $(P\Gamma T\Gamma)^{n-1}P = \{0\}$.

Remark 4.11: A strongly nilpotent ternary Γ-ideal of a ternary Γ-semiring is strongly semi-nilpotent.

Theorem 4.12: If P is a strongly semi-nilpotent right ternary Γ-ideal of T then P ⊆ J (T).

Proof: Suppose on the contrary that $P \not\subseteq J(T) = \bigcap_{M\in\Delta} A_T(M)$, where Δ is the set of all irreducible right ternary *T*Γ-semimodules. Then there exist M ∈Δ such that $P \not\subseteq A_T(M)$.

This implies that $M\Gamma T\Gamma P \neq \{0\}$ and $M\Gamma P\Gamma T \neq \{0\}$, by the definition of $A_T(M)$.

Since P is strongly semi-nilpotent, there exist a positive integer *n* such that $(P\Gamma T\Gamma)^{n-1}P \subseteq Z(T) \Rightarrow$ for $p_i\in$ P (i = 1, 2… n), $t_i\in$ T (i = 1, 2,,,,, n -1),

$$p_1\Gamma t_1\Gamma p_2\Gamma t_2\Gamma.....\Gamma p_{n-1}\Gamma t_{n-1}\Gamma p_n + z = z \text{ For some } z\in \text{T}$$

$$\Rightarrow m\Gamma t\Gamma(p_1\Gamma t_1\Gamma p_2\Gamma t_2\Gamma.....\Gamma p_{n-1}\Gamma t_{n-1}\Gamma p_n) + m\Gamma t\Gamma z = m\Gamma t\Gamma z \text{ For some } m\in \text{M and for all } t\in \text{T.}$$

Again, we further we deduce that $m\Gamma t\Gamma(p_1\Gamma t_1\Gamma p_2\Gamma t_2\Gamma.....\Gamma p_{n-1}\Gamma t_{n-1}\Gamma p_n) = 0$ for all $m\in$ M and for all $t\in$ T. Since M is additively cancellative. This shows that $M\Gamma T\Gamma(P\Gamma T\Gamma)^{n-1}P = \{0\}$. If the above relation hold for all *n*, then in particular it holds for *n* = 1 and in this case $M\Gamma T\Gamma P = \{0\}$ which is a contradiction, since $M\Gamma T\Gamma P \neq \{0\}$ by hypothesis.

Thus there exist $m\in$ M and a positive integer *k* such that

$$m\Gamma T\Gamma(P\Gamma T\Gamma)^{k-1}P \neq \{0\} \text{ And } m\Gamma T\Gamma(P\Gamma T\Gamma)^{k}P = \{0\}$$

Let $u(\neq 0) \in m\Gamma T\Gamma(P\Gamma T\Gamma)^{k-1}P \subseteq M$. Since M is irreducible, hence it is semi-irreducible by corollary 4.4, and hence by lemma 4.3, for $m \in$ M there exist

$a_i, b_i, c_j, d_j \in T, \alpha_i \beta_i, \alpha_j, \beta_j \in \Gamma(1 \le i \le p, 1 \le j \le q; p, q$ Are positive integers) such that

$$m + \sum_{i=1}^{p} u\alpha_i a_i \beta_i b_i = \sum_{j=1}^{q} u\alpha_j c_j \beta_j d_j .$$

Hence, we have shown that $m\alpha t\beta r + \sum_{i=1}^{p} u\alpha_i a_i \beta_i b_i \alpha t\beta r = \sum_{j=1}^{q} u\alpha_j c_j \beta_j d_j \alpha t\beta r$ for all $t \in T$ and for all $r \in P$.

Since $\sum_{i=1}^{p} u\alpha_i a_i \beta_i b_i \alpha t\beta r$, $\sum_{j=1}^{q} u\alpha_j c_j \beta_j d_j \alpha t\beta r \in M\Gamma T\Gamma(P\Gamma T\Gamma)^{n-1}P\Gamma T\Gamma T\Gamma T\Gamma P$

$$\subseteq M\Gamma T\Gamma(P\Gamma T\Gamma)^{n-1}P\Gamma T\Gamma P = m\Gamma T\Gamma(P\Gamma T\Gamma)^{k}P = \{0\}$$

We have $m\Gamma t\Gamma r = 0$ for all $t \in$ T and $r \in$ P. This leads to MΓTΓP = {0}, which is again a contradiction. This completes the proof of the theorem.

By theorem 4.12 and remark 4.11, we obtain the following corollary.

Corollary 4.13: If a ternary -semiring T is Jacobson semisimple then T does not contain any non-zero strongly semi-nilpotent right ternary -ideal and hence T does not contain any non-trivial strongly nilpotent right ternary -ideal.

CONCLUSIONS

In this paper mainly we start the study of primitive ternary Γ-semiring and Jacobson radicals, in ternary Γ-semirings. We characterize them.

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